

# On traces of operators, associated with actions of compact Lie groups

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## Abstract

Given a pair  $(M, X)$ , where  $X$  is a smooth submanifold in a closed smooth manifold  $M$ , we study the operation, which takes each operator  $D$  on the ambient manifold to a certain operator on the submanifold. The latter operator is called the trace of  $D$ . More precisely, we study traces of operators, associated with actions of compact Lie groups on  $M$ . We show that traces of such operators are localized at special submanifolds in  $X$  and study the structure of the traces on these submanifolds.

## 1 Statement of the problem

Let  $(M, X)$  be a pair, where  $M$  is a smooth manifold and  $X$  is a smooth submanifold in  $M$ . The embedding of  $X$  in  $M$  is denoted by  $i : X \rightarrow M$ .

Given an operator  $D$  acting on functions on  $M$ , we define its trace on  $X$  as

$$i^* D i_* : H^s(X) \longrightarrow H^{s-d-\nu}(X), \quad (1)$$

where  $i^* : H^s(M) \rightarrow H^{s-\nu/2}(X)$  is the boundary operator induced by the embedding  $i : X \rightarrow M$ , while  $i_* : H^s(X) \rightarrow H^{s-\nu/2}(M)$  is the coboundary operator conjugate to  $i^*$ ,  $d$  is the order of  $D$ ,  $\nu$  is the codimension of  $X$  in  $M$ . We denote the trace by  $i^!(D)$ .

The aim of this paper is to study traces of  $G$ -operators on  $M$ . This class of operators is associated with a smooth action of a compact Lie group  $G$  on  $M$ . Then a  $G$ -operator is defined by an integral of the form

$$D = \int_G D_g T_g dg, \quad (2)$$

where  $T_g$  stands for the shift operator

$$T_g u = g^{-1*} u$$

induced by a diffeomorphism  $g \in G$ ,  $dg$  is the Haar measure on  $G$ ,  $\{D_g\}$  is a smooth family of pseudodifferential operators ( $\psi$ DO below) parametrized by  $g \in G$ .

The notion of trace in the smooth theory, i.e., in the situation, when the group is trivial, appeared in the works [1–3]. There it was shown, in particular, that the trace of a pseudodifferential operator is a pseudodifferential operator on  $X$ . This fact made it possible to apply pseudodifferential operators in the theory of Sobolev problems.

If a nontrivial group  $G$  acts on  $M$  and we consider  $G$ -operators (2) (see [4–6]), the situation is completely different. Namely, the trace of a  $G$ -operator on a submanifold is an operator localized at a certain submanifold in  $X$  (for instance, it can be localized at a point, see [7, 8]). Moreover, the trace is not a pseudodifferential operator and is an operator of fundamentally new nature, for instance, it can be localized at a point (this can happen at the isolated fixed point of the group action), and to describe such operators, one needs not only Fourier transform (as for  $\psi$ DO's), but also Mellin transform.

The present paper is devoted to studying this new class of operators. The main result is a localization theorem, which describes the set, on which the operator is localized. This theorem is given in Section 2. In subsequent sections we consider a number of examples, which clarify the situation both from the geometric (Section 3) and analytic (Section 4) points of view. Namely, we consider the situation when  $X$  is invariant with respect to the group action (in this case the trace of a  $G$ -operator is again a  $G$ -operator on  $X$ ), we also consider noninvariant submanifolds  $X$ , in which case the trace is localized on a lower-dimensional submanifold.

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## 2 Localization theorem

As we stressed in the introduction, the trace of a  $G$ -operator  $D$  on a submanifold  $X \subset M$  is generally localized at a certain subset in  $X$ . To describe this set, we consider the following two closed subsets in  $X$ :

1. The set of points whose orbits are contained in  $X$

$$X_G = \{x \in X \mid Gx \subset X\}, \quad \text{where } Gx \text{ denotes the orbit of } x;$$

2. The set of points whose orbits touch  $X$

$$\tilde{X}_G = \{x \in X \mid T_x(Gx) \subset T_x X\}.$$

Obviously, we have  $X_G \subset \tilde{X}_G$ .

Below we assume that the following condition is satisfied:

**Condition 2.1.** For each closed subset  $Z \subset \tilde{X}_G \setminus X_G$  there exists a family of open subsets  $U_\varepsilon \subset X$ , which contract to  $Z$  as  $\varepsilon \rightarrow 0$ , and such that the volume of the open set in  $G$

$$G_\varepsilon = \{g \in G \mid gU_\varepsilon \cap X \neq \emptyset\}$$

tends to zero as  $\varepsilon \rightarrow 0$ .

Condition 2.1 is easy to check in examples (see below). The meaning of this condition is clear: an element  $g \in G$  takes the neighborhood  $U_\varepsilon$  to a set disjoint with  $X$  for almost all  $g$  (roughly speaking, if an orbit is not contained in  $X$ , then it is outside of  $X$  almost everywhere).

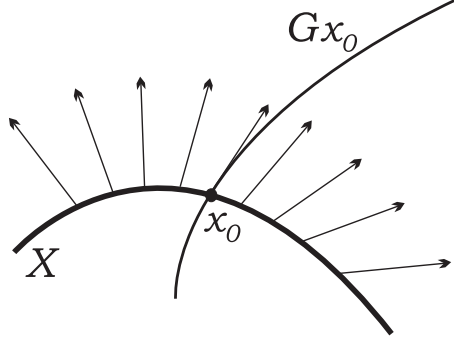


Figure 1: A neighborhood of a point  $x_0$ , at which the orbit is not tangent to  $X$ .

**Definition 2.1.** An operator  $A$  is *localized on a subset*  $Y \subset X$ , if all compositions  $A\varphi$  are compact, whenever  $\varphi$  is a smooth function equal to zero in a neighborhood of  $Y$ .

**Theorem 2.1.** (on localization) *The trace of a  $G$ -operator  $D$  on a submanifold  $X$  is localized on the subset  $X_G \subset X$ . In particular, if  $X_G$  is empty, then the trace is a compact operator.*

*Proof.* 1. Consider the operator

$$Di_* = \int_G D_g T_g i_* dg. \quad (3)$$

Note that there is no restriction operator in this expression. We claim that this operator (and hence the trace  $i^* Di_*$ ) is localized on the subset  $\tilde{X}_G$ .

To prove this statement, we fix an arbitrary point  $x_0 \in X \setminus \tilde{X}_G$  and show that the operator (3) is compact on the subspace of functions supported in a neighborhood of  $x_0$ . To this end, we decompose the integral (3) over  $G$  into a finite sum of integrals over small neighborhoods and move the shift operator at an arbitrary point in each neighborhood outside the integral sign. This enables us without loss of generality to pass to an operator of the form (3), where the integration is carried out over a small neighborhood of the identity in  $G$ . Further, we identify this neighborhood with a neighborhood  $U$  of zero in the Lie algebra denoted by  $\mathcal{G}$ . By the assumption,  $x_0 \in X \setminus \tilde{X}_G$ , i.e., at  $x_0$  the orbit  $Gx_0$  is not tangent to  $X$ . Hence, there exists a vector  $h \in \mathcal{G}$  such that the corresponding vector field on  $M$  is not tangent to  $X$  at  $x_0$  (and also in a neighborhood of this point by continuity, see Fig. 1).

Then the integral (3) (recall that this integral is actually only over  $U$ ) can be considered as an iterated integral: first along  $h$  and then over directions transverse to  $h$ . Let us show that the integral in the direction of  $h$

$$\int_{-\varepsilon}^{\varepsilon} D'_a T_{\exp(ah)} i_* da, \quad \text{where } D'_a \equiv D_{\exp a} \text{ for simplicity,} \quad (4)$$

is a compact operator.

**Lemma 2.1.** *The operator (4) acts continuously in the spaces  $H^s(X) \rightarrow H^{s-(d+(\nu-1)/2)}(M)$ . In particular, it is compact as an operator of order  $d + \nu$ .*

*Proof.* The idea of the proof is that integration along  $h$  is an integration in a transverse direction to  $X$ , and, when we perform it, there disappears *one* of the  $\delta$ -functions, which enter in the coboundary operator, hence, we get the summand  $-1/2$  in the formula given above. Let us give a detailed proof.

1. In a neighborhood of  $x_0$  we introduce coordinates  $x, y, t$  in  $M$ , in which  $X$  is determined by the equations  $y = 0, t = 0$ , while the action of an element  $\exp(ah)$  with  $|a|$  small on  $M$  is equal to  $\exp(ah)(x, y, t) = (x, y, t + a)$ . The operator (4) is denoted by  $A$  for simplicity and we write it explicitly as

$$\begin{aligned} Au &= \int_{-\varepsilon}^{\varepsilon} D'_a u(x) \delta(y) \delta(t - a) da = \\ &= \iiint e^{i(x\xi + y\eta + t\tau)} \left[ \int_{-\varepsilon}^{\varepsilon} e^{-i\tau a} \sigma(D'_a)(x, y, t, \xi, \eta, \tau) da \right] \tilde{u}(\xi) d\xi d\eta d\tau. \end{aligned} \quad (5)$$

Here the coordinates  $\xi, \eta, \tau$  are dual to  $x, y, t$ , while  $\sigma(D'_a)$  is the symbol of the pseudodifferential operator  $D'_a$ , and  $\tilde{u}(\xi)$  stands for the Fourier transform of  $u(x)$

2. Given  $u \in H^s(X)$ , we have to show that  $Au \in H^{s'}(M)$ , where  $s' = s - d - \nu/2 + 1/2$ . Indeed, integrating by parts the expression in the square brackets in (5), we obtain the estimate

$$\left| \int_{-\varepsilon}^{\varepsilon} e^{-i\tau a} \sigma(D_a)(x, y, t, \xi, \eta, \tau) da \right| \leq C(1 + |\tau|)^{-1}(1 + |\xi| + |\eta| + |\tau|)^d. \quad (6)$$

Here  $C$  is some constant. Hence, we have

$$\begin{aligned} \|Au\|_{s'}^2 &\leq C \iiint \tilde{u}(\xi)^2 (1 + |\tau|)^{-2} (1 + |\xi| + |\eta| + |\tau|)^{2d+2s'} d\xi d\eta d\tau \leq \\ &\leq C \iint \tilde{u}(\xi)^2 (1 + |\tau|)^{-2} (1 + |\xi| + |\tau|)^{2d+2s'+\nu-1} d\xi d\tau \leq C \int \tilde{u}(\xi)^2 (1 + |\xi|)^{2s} d\xi = \|u\|_s^2. \end{aligned} \quad (7)$$

Here the first inequality follows from the properties of the Fourier transform, the second is obtained by integration over  $\eta$  and using the spherical coordinates  $\eta = (1 + |\xi| + |\tau|)r\omega$ ; the third inequality is obtained using the change of variables  $1 + |\tau| = |\xi|p$  as follows

$$\begin{aligned} \int_{\mathbb{R}} (1 + |\tau|)^{-2} (1 + |\xi| + |\tau|)^{2d+2s'+\nu-1} d\tau &= 2|\xi|^{-2+2d+2s'+\nu-1+1} \int_{|\xi|^{-1}}^{\infty} p^{-2} (1 + p)^{2d+2s'+\nu-1} dp \\ &= C|\xi|^{s-1} \times \begin{pmatrix} |\xi| & \text{if } |\xi| > 1 \\ |\xi|^{1-s} & \text{if } |\xi| < 1 \end{pmatrix} \leq C(1 + |\xi|)^{2s}. \end{aligned} \quad (8)$$

The proof of the lemma is now complete.  $\square$

2. Let us show that the trace  $i^*Di_*$  is indeed supported on the subset  $X_G \subset \tilde{X}_G$ . To this end, we consider an arbitrary function  $\varphi \in C^\infty(X)$ , which vanishes in a neighborhood of  $X_G$ . We claim that the operator

$$i^*Di_*\varphi : H^s(X) \longrightarrow H^{s-d-\nu}(X)$$

is compact. Indeed, let us use Condition 2.1 and take  $Z$  equal to  $\tilde{X}_G \cap \text{supp } \varphi$ .

Consider the decomposition

$$i^*Di_*\varphi = \int_{G_\varepsilon} i^*D_gT_gi_*\varphi dg + \int_{G \setminus G_\varepsilon} i^*D_gT_gi_*\varphi dg. \quad (9)$$

Here the first integral has a small norm (since  $\text{vol}(G_\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ), while the second integral is, on the one hand, (by Item 1 above) supported on  $Z$ , and on the other hand, on this set it is compact by locality (indeed, if  $x \in U_\varepsilon$  and  $g \in G \setminus G_\varepsilon$ , then  $gx \in M \setminus X$ , hence  $i^*D_gT_gi_*\varphi$  is compact). As  $\varepsilon \rightarrow 0$ , it follows from the decomposition (9) into a sum of an operator with small norm and a compact operator that the operator on the left hand side of the equality is compact.

The proof of the localization theorem is now complete.  $\square$

### 3 Examples (geometry)

Let us give examples of manifolds and group actions and apply the localization theorem (Theorem 2.1) to them.

**Rotations of lines in the plane.**  $M = \mathbb{R}^2, X = \mathbb{R}^1$ , while  $G = \mathbb{S}^1$  is the group of rotations around the origin. Here there are two possibilities, depending on whether the line  $X$  passes through the origin or not. If the line passes through the origin, then the traces of operators are supported at the fixed point  $X_G = \tilde{X}_G = \{A\}$  (see Fig. 2, Item 1). If the line does not pass through the fixed point, then the traces of operators are all compact. More precisely, each such line has a unique point  $\tilde{X}_G = A$ , whose orbit touches  $X$ , while there are no fixed points:  $X_G = \emptyset$  (see Fig. 2, Item 2). Meanwhile, Condition 2.1 is satisfied (as  $U_\varepsilon$  we can take an interval of length  $\varepsilon$  with center at  $A$ ).

**Shifts of the plane.**  $M = \mathbb{R}^2, X = \mathbb{S}^1$ , while  $G = \mathbb{R}$  is the group of translations (see Fig. 2, Item 3). This example is similar to the previous one, namely, the orbits are tangent to  $X$  at the points denoted by  $A$  and  $B$ , while there are no fixed points:  $\tilde{X}_G = \{A\} \cup \{B\}$ ,  $X_G = \emptyset$ . Hence, traces of operators on  $X$  are compact (Condition 2.1 is satisfied in this case). On Fig. 2, Item 4 we consider an example, in which our Condition 2.1 is not satisfied (here  $X_G = \emptyset$ , while  $\tilde{X}_G$  consists of the horizontal intervals).

**Rotations of lines in  $\mathbb{R}^3$ .**  $M = \mathbb{R}^3, X = \mathbb{R}^1$ , while  $G = \mathbb{S}^1$  is the group of rotations about the  $OZ$ -axis, while the line  $X$  and the axis of rotation intersect and form angle  $\alpha$  (see Fig. 3, Item 1). In this case traces of  $G$ -operators are localized at the intersection of the two lines, except the case, when  $\alpha = 0$  (more precisely, for  $\alpha \neq 0$  we have  $\tilde{X}_G = X_G = \{A\}$ ).

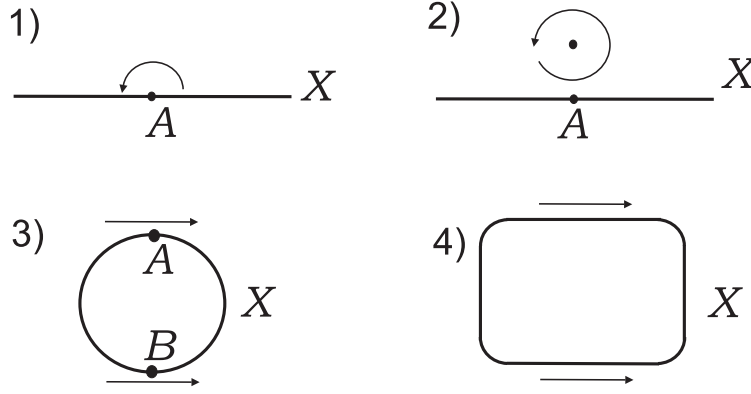


Figure 2: Rotations and shifts of the plane.

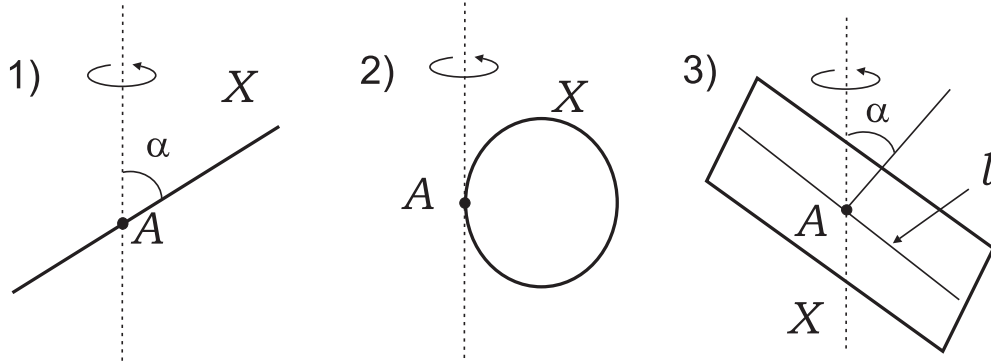


Figure 3: Rotations in  $\mathbb{R}^3$ .

**Rotations of circles in  $\mathbb{R}^3$ .**  $M = \mathbb{R}^3$ ,  $X = \mathbb{S}^1$ , while  $G = \mathbb{S}^1$  is the group of rotations about the  $OZ$ -axis and the circle  $X$  touches the axis of rotation (see Fig. 3, Item 2). In this case the trace of operators is localized at the point, where the circle meets the axis of rotation. This degenerate case is interesting, since the trace is also localized at a one-point set and there arises the question about the nature of this operator.

**Rotations of planes in  $\mathbb{R}^3$ .**  $M = \mathbb{R}^3$ ,  $X = \mathbb{R}^2$ ,  $G = \mathbb{S}^1$  is the group of rotations around the  $OZ$ -axis, while the normal to the plane  $X$  and the axis of rotation form an angle equal to  $\alpha$  (see Fig. 3, Item 3). In this case traces of  $G$ -operators are localized at the point of intersection of the plane with the axis of rotation (more precisely,  $\tilde{X}_G$  is the line  $l$ , while  $X_G = \{A\}$ ).

**Rotations and shifts of spheres in  $\mathbb{R}^3$ .**  $M = \mathbb{R}^3$ ,  $X = \mathbb{S}^2$ , while  $G = \mathbb{S}^1 \times \mathbb{R}^1$  is the group generated by rotations about  $OZ$  and shifts along this axis. The sphere  $X$  has center at the origin (see Fig. 4, Item 1). In this case traces of  $G$ -operators are compact. Indeed, the orbits touch the spheres at the equator  $\tilde{X}_G = \mathbb{S}^1$ , while none of the orbits is contained in the sphere and thus  $X_G = \emptyset$  (Condition 2.1 is satisfied in this case).

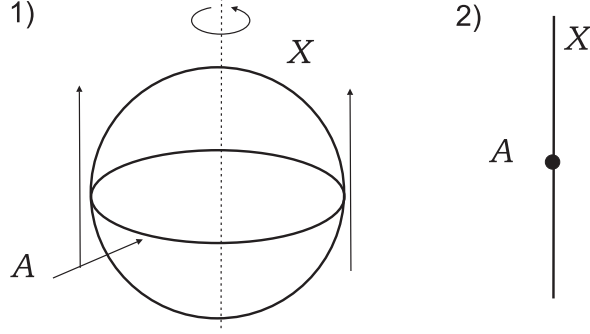


Figure 4: Actions of Lie groups 1)  $G = \mathbb{S}^1 \times \mathbb{R}$  and 2)  $SO(3)$  in  $\mathbb{R}^3$ .

**Arbitrary rotations of a line in  $\mathbb{R}^3$ .**  $M = \mathbb{R}^3$ ,  $X = \mathbb{R}^1$ , while  $G = SO(3)$  is the group of all rotations about the origin. Here the line  $X$  passes through the origin. In this case traces of  $G$ -operators are localized at the origin (see Fig. 4, Item 2).

## 4 Examples (analysis)

In this section we consider examples and study the nature of traces of  $G$ -operators in a neighborhood of the set, on which they are localized.

**Example 1.** Let a submanifold  $X \subset M$  be  $G$ -invariant, i.e. we have  $X = X_G = \tilde{X}_G$ . In this case traces of  $G$ -operators on  $M$  are localized on the entire manifold  $X$  by Theorem 2.1. It turns out, that in this invariant situation the trace is actually, a  $G$ -operator on  $X$  with respect to the restriction of the action of  $G$  on  $X$  and the corresponding shift operators

$$T'_g : H^s(X) \longrightarrow H^s(X), \quad T'_g u(x) = u(g^{-1}x).$$

More precisely, the following statement is true.

**Proposition 4.1.** *The trace of a  $G$ -operator  $D$  (see (2)) on a  $G$ -invariant submanifold  $X$  is a  $G$ -operator on  $X$ , i.e., we have*

$$i^!(D) = \int_G D'_g T'_g dg, \quad (10)$$

where  $D'_g$  is a smooth family of pseudodifferential operators on  $X$ .

*Proof.* Let us compute the trace of the integrand in (2). We have

$$i^!(D_g T'_g)u(x) = i^* D_g T'_g (u(x) \otimes \delta_X) = i^* D_g (T'_g u(x) \otimes \delta_X) = (i^* D_g i_*)(T'_g u(x)) = i^!(D_g) T'_g u(x).$$

Here we used the fact that  $G$  is compact and chose a  $G$ -invariant  $\delta$ -function supported on  $X$ . Hence, integration over  $G$  gives the desired equality (10), where  $D'_g = i^!(D_g)$  is a pseudodifferential operator on the submanifold.

The proof of the proposition is now complete.  $\square$

Of course, a submanifold is not  $G$ -invariant in general. Below we study traces in such situations.

**Example 2.** Let  $X \subset \mathbb{R}^3$  be the plane

$$-x \sin \alpha + z \cos \alpha = 0.$$

A basis in this plane is given by the vectors  $e_1 = (\cos \alpha, 0, \sin \alpha)$ ,  $e_2 = (0, 1, 0)$ . A normal vector is equal to  $e_3 = (-\sin \alpha, 0, \cos \alpha)$ . The coordinates in the basis  $e_1, e_2, e_3 \in \mathbb{R}^3$  are denoted by  $(u, v, w)$ . These coordinates are related with the coordinates  $x, y, z$  as

$$(x, y, z) = (u \cos \alpha - w \sin \alpha, v, u \sin \alpha + w \cos \alpha). \quad (11)$$

In  $\mathbb{R}^3$ , we consider the scalar  $G$ -operator

$$D = \Delta^{-1} \int_{\mathbb{S}^1} T_\varphi d\varphi, \quad (12)$$

where  $\Delta$  stands for the Laplacian, while  $T_\varphi$  is the shift operator associated with the group of rotations about the  $OZ$ -axis:

$$(T_\varphi f)(x, y, z) = f(x \cos \varphi - y \sin \varphi, x \sin \varphi + y \cos \varphi, z).$$

Let us study the trace

$$i^!(D) = i^* \left( \Delta^{-1} \int_{\mathbb{S}^1} T_\varphi d\varphi \right) i_* : H^s(X) \rightarrow H^{s+1}(X), \quad s \in (-1, 0), \quad (13)$$

of  $D$  on  $X$ .

**Proposition 4.2.** *The trace of operator (12) is localized at the point  $(0, 0, 0)$  of intersection of the plane  $X$  with the axis of rotation.*

*Proof.* In this case a direct computation shows that  $\tilde{X}_{\mathbb{S}^1} = \{y = 0, -x \sin \alpha + z \cos \alpha = 0\}$  is a line, while  $X_{\mathbb{S}^1} = \{(0, 0, 0)\}$  is a point. Hence, by Theorem 2.1 the trace is localized at  $(0, 0, 0)$ .  $\square$

So, the trace is localized at the fixed point of the group action. Let us study its structure by freezing the coefficients of the operator at this point. Below, it will be more convenient to work with zero-order operators. We make reduction to this case by taking products of our operator with appropriate powers of the Laplacian  $\Delta_X$  on  $X$ , i.e., from operator (12) we pass to the operator

$$\Delta_X^{1/2} i^* \left( \Delta^{-1} \int_{\mathbb{S}^1} T_\varphi d\varphi \right) i_* : H^s(\mathbb{R}^2) \rightarrow H^s(\mathbb{R}^2). \quad (14)$$

A direct computation shows that in the dual space with respect to the Fourier transform operator (14) is written as an integral operator as

$$\begin{aligned} f(s, t) \longmapsto & (u^2 + v^2)^{1/2} \int_{\mathbb{R}} dw \int_0^{2\pi} \frac{d\varphi}{u^2 + v^2 + w^2} \times \\ & f \left( u(\cos^2 \alpha \cos \varphi + \sin^2 \alpha) + v \cos \alpha \sin \varphi + w \sin \alpha \cos \alpha (1 - \cos \varphi), -u \cos \alpha \sin \varphi + v \cos \varphi + w \sin \alpha \sin \varphi \right) : \\ & : \tilde{H}^s(\mathbb{R}_{s,t}^2) \longrightarrow \tilde{H}^s(\mathbb{R}_{u,v}^2). \end{aligned} \quad (15)$$



Here we used the fact that operators on the physical space are transformed to the following operators on the dual space:

- the coboundary operator  $i_*$  is transformed to the operator

$$\pi^* f(x, y, z) = f(x \cos \alpha + z \sin \alpha, y)$$

where  $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  denotes the projection;

- the rotation operator  $T_\varphi$  is transformed to the rotation operator

$$\tilde{T}_\varphi f(x, y, z) = f(x \cos \varphi + y \sin \varphi, -x \sin \varphi + y \cos \varphi, z);$$

- the boundary operator  $i^*$  is transformed to the operator of integration with respect to  $w$ :

$$\pi_* f(u, v) = \int_{\mathbb{R}} f(u \cos \alpha - w \sin \alpha, v, u \sin \alpha + w \cos \alpha) dw$$

(here we used the change of variables (11));

- replacing the operators in the composition (14) by the corresponding operators on the dual space, we obtain precisely the operator (15);
- the space  $\tilde{H}^s(\mathbb{R}_{u,v}^2)$  is the closure of the set of smooth compactly-supported functions with respect to the norm

$$\|f\|_s^2 = \int_{\mathbb{R}^2} |f(u, v)|^2 (1 + u^2 + v^2)^s du dv.$$

Below we use polar coordinates on  $X$ :

$$s = \rho \cos \psi, t = \rho \sin \psi \quad \text{and} \quad u = r \cos \omega, v = r \sin \omega.$$

In the integral in (15) we make the following change of variables  $(w, \varphi) \mapsto (\rho, \psi)$ :

$$\begin{aligned} u(\cos^2 \alpha \cos \varphi + \sin^2 \alpha) &+ v \cos \alpha \sin \varphi &+ w \sin \alpha \cos \alpha (1 - \cos \varphi) &= \rho \cos \psi, \\ -u \cos \alpha \sin \varphi &+ v \cos \varphi &+ w \sin \alpha \sin \varphi &= \rho \sin \psi. \end{aligned} \tag{16}$$

It turns out that this change of variables is one-to-one, while the inverse change is equal to (cumbersome computations are omitted):

$$\begin{aligned} \tan \frac{\varphi}{2} &= \frac{\rho \cos \psi - u}{(v + \rho \sin \psi) \cos \alpha}, \\ w &= u \cot \alpha - \frac{v \cot \varphi}{\sin \alpha} + \frac{\rho \sin \psi}{\sin \alpha \sin \varphi} = \\ &= \frac{u^2(1 - 2 \cos^2 \alpha) - 2\rho u \cos \psi \sin^2 \alpha + \rho^2(\cos^2 \psi + \sin^2 \psi \cos^2 \alpha) - v^2 \cos^2 \alpha}{2(\rho \cos \psi - u) \cos \alpha \sin \alpha} = \\ &= \frac{\sin^2 \alpha (\rho \cos \psi - u)^2 + \cos^2 \alpha (\rho^2 - r^2)}{2(\rho \cos \psi - u) \cos \alpha \sin \alpha}. \end{aligned} \tag{17}$$

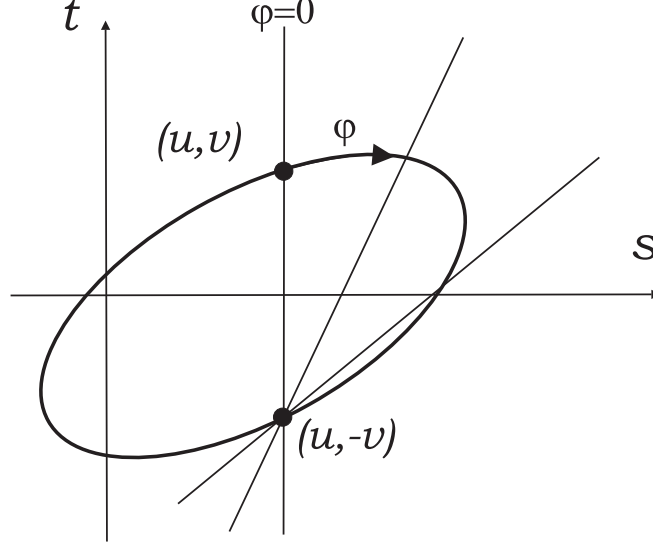


Figure 5: Change of variables  $(w, \varphi) \mapsto (\rho, \omega)$ .

Let us describe the geometric meaning of this change of variables. In the  $(s, t)$ -plane we have an ellipse (defined parametrically in terms of  $\varphi$ ):

$$s = u(\cos^2 \alpha \cos \varphi + \sin^2 \alpha) + v \cos \alpha \sin \varphi, \quad t = -u \cos \alpha \sin \varphi + v \cos \varphi.$$

As  $\varphi$  increases, the corresponding point goes around the ellipse clockwise, starting from the point  $(u, v)$  (see Fig. 5). Further, for each point of this ellipse (i.e., for a given  $\varphi$ ) equations (16) define a line in the  $(s, t)$ -plane, which passes through this point with parameter  $w$  along the line (this line degenerates to a point for  $\varphi = 0$ ). A direct computation shows that this line passes through the above mentioned point of the ellipse and the point with the coordinates  $(u, -v)$ .

Clearly, the mapping  $(w, \varphi) \mapsto (\rho, \omega)$  is a diffeomorphism, except at the points, which lie on the vertical line passing through the point  $(u, -v)$  of the ellipse.

Making the change of variables (16), we rewrite the product of the differentials in (15) in the new coordinates as

$$d\varphi dw = \frac{\rho d\rho d\psi}{\cos \alpha \sin \alpha ((1 - \cos \varphi)(w \sin \alpha - u \cos \alpha) + v \sin \varphi)} = \frac{\rho d\rho d\psi}{\sin \alpha (\rho \cos \psi - u)}. \quad (18)$$

Substituting (16) and (18) in the integral operator (15), we rewrite this integral operator as

$$f(s, t) \mapsto r \int_0^\infty d\rho \int_0^{2\pi} \frac{\rho d\psi}{(r^2 + w^2) \sin \alpha |\rho \cos \psi - u|} f(\rho \cos \psi, \rho \sin \psi) = \int_0^\infty K(\rho/r) f(\rho) \frac{d\rho}{\rho}, \quad (19)$$

where  $K(\rho)$  is a family of integral operators on the circle equal to

$$\begin{aligned} (K(\rho)f)(\omega) &= \int_0^{2\pi} \frac{\rho^2}{(1+w^2)\sin\alpha|\rho\cos\psi - \cos\omega|} f(\psi) d\psi = \\ &= \int_0^{2\pi} \frac{4\sin\alpha\cos^2\alpha\rho^2|\rho\cos\psi - \cos\omega|}{4\cos^2\alpha\sin^2\alpha(\rho\cos\psi - \cos\omega)^2 + (\sin^2\alpha(\rho\cos\psi - \cos\omega)^2 + \cos^2\alpha(\rho^2 - 1))^2} f(\psi) d\psi. \end{aligned} \quad (20)$$

Since the operator-function  $K(\rho/r)$  in (19) is homogeneous of degree zero with respect to the pair of its arguments, operator (19) is nothing but the Mellin convolution in the variable  $r$ . Hence, it is algebraized if we apply the Mellin transform  $\mathcal{M}_{\rho \rightarrow p}$ . Here by algebraization we mean that the operator is written as an operator of multiplication by the function

$$\widehat{K}(p) = \mathcal{M}_{\rho \rightarrow p} K(\rho) = \int_0^\infty \rho^p K(\rho) \frac{d\rho}{\rho}. \quad (21)$$

Necessary properties of this operator-function are described in the following two lemmas.

**Lemma 4.1.** *The operator-function  $K(\rho)$  ranges in integral operators with smooth kernel for all  $\rho > 0$  and  $\rho \neq 1$ , and its operator norm in the space  $L^2(\mathbb{S}^1)$  has the following estimates*

$$\|K(\rho)\| = \begin{cases} O(\rho^2), & \text{if } \rho < 1/2, \\ O(|\rho - 1|^{-1/2}), & \text{if } 1/2 < \rho < 2, \\ O(\rho^{-1}), & \text{if } \rho > 2. \end{cases} \quad (22)$$

*Proof.* 1. The singularities of the Schwarz kernel of the operator (20) correspond to the zeroes of the denominator. Since this denominator is a sum of squares, the singularities of the denominator are determined from the equations

$$\rho\cos\psi - \cos\omega = 0, \quad \rho^2 - 1 = 0,$$

which are equivalent to  $\rho = 1, \psi = \pm\omega$ . This implies that for  $\rho \neq 1$  the denominator has no zeroes, hence, the Schwarz kernel is smooth. The first statement in the lemma is now proved.

2. Estimates of the integral kernel and the operator norm as  $\rho \rightarrow \infty$  and  $\rho \rightarrow 0$  are obtained similarly. Namely, as  $\rho \rightarrow \infty$  the numerator in (20) is equal to  $O(\rho^3)$  and the denominator has a lower bound  $\geq C\rho^4$ , which gives us the desired estimate. Finally, as  $\rho \rightarrow 0$  the numerator is of order  $O(\rho^2)$ , while the denominator is separated from zero, which gives the desired estimate.

3. It remains to estimate the norm of operator  $K(\rho)$  as  $\rho \rightarrow 1$ . So, we consider  $\rho$  close but not equal to 1. To estimate the norm of the integral operator  $K(\rho)$ , we use the Schur test (e.g., see [9]) and estimate the integrals

$$\int_{\mathbb{S}^1} |K(\rho, \omega, \psi)| d\omega, \quad \int_{\mathbb{S}^1} |K(\rho, \omega, \psi)| d\psi$$

of the kernel  $K(\rho, \omega, \psi)$  uniformly in  $\omega, \psi$ . Let us estimate the first of the integrals (the second is estimated similarly). We have

$$\begin{aligned} \int_{\mathbb{S}^1} |K(\rho, \omega, \psi)| d\omega &\leq C \int_{\psi-\varepsilon}^{\psi+\varepsilon} \frac{|\rho \cos \psi - \cos \omega| d\omega}{(\rho \cos \psi - \cos \omega)^2 + (\tan^2 \alpha (\rho \cos \psi - \cos \omega)^2 + (\rho^2 - 1))^2} \leq \\ &\leq C \int_{-\varepsilon_1}^{\varepsilon_2} \frac{|(\rho - 1) \cos \psi - t| dt}{\left[ ((\rho - 1) \cos \psi - t)^2 + (\tan^2 \alpha ((\rho - 1) \cos \psi - t)^2 + (\rho^2 - 1))^2 \right] \sqrt{|\sin^2 \psi - 2t \cos \psi - t^2|}} \end{aligned} \quad (23)$$

(for some numbers  $\varepsilon_1, \varepsilon_2 \geq 0$ , which are bounded uniformly in  $\psi$  and  $\rho$ ). In the last inequality in (23) we reduced our integral to an integral over a small neighborhood of the point  $\psi$ , since the integrand is uniformly bounded whenever  $|\omega \pm \psi| > \varepsilon$  and is an even function. Then in the second inequality we made the change of variable  $\omega \mapsto t$ :

$$\cos \omega = \cos \psi + t, \quad d\omega = \frac{\pm dt}{\sqrt{|\sin^2 \psi - 2t \cos \psi - t^2|}}.$$

Then in the latter integral in (23) we make the change of variable  $t = |\rho - 1|\tau$ ; then we obtain

$$\begin{aligned} \int_{\mathbb{S}^1} |K(\rho, \omega, \psi)| d\omega &\leq \\ &\leq C \int_{-\varepsilon_1 |\rho-1|^{-1}}^{\varepsilon_2 |\rho-1|^{-1}} \frac{|\cos \psi \mp \tau|}{\left[ (\cos \psi \mp \tau)^2 + (\tan^2 \alpha (\rho - 1)(\cos \psi \mp \tau)^2 + (\rho + 1))^2 \right]} \times \\ &\quad \times \frac{d\tau}{\sqrt{|\sin^2 \psi - 2\tau |\rho - 1| \cos \psi - \tau^2 |\rho - 1|^2|}} \leq \\ &\leq C \int_{-\varepsilon_1 |\rho-1|^{-1}}^{\varepsilon_2 |\rho-1|^{-1}} \frac{d\tau}{(|\tau| + 1) \sqrt{(|\rho - 1|\tau + \cos \psi + 1)(|\rho - 1|\tau + \cos \psi - 1)|}} \end{aligned} \quad (24)$$

The second inequality here follows from the fact that the numerator is  $O(|\tau| + 1)$ , while the expression in square brackets in the denominator is nonzero, and  $\geq \tau^2$  at infinity. The last integral in (24) admits the estimate

$$\leq C \int_{-\varepsilon_1 |\rho-1|^{-1}}^{\varepsilon_2 |\rho-1|^{-1}} \frac{d\tau}{|\tau| + 1} \leq C \ln |\rho - 1|^{-1},$$

provided that  $|\cos \psi \pm 1| > \varepsilon_1, \varepsilon_2$ . Let us now consider the case, when one of the numbers  $\cos \psi \pm 1$  is small. For definiteness, we consider the case, when  $\psi$  is close to zero (the case, when  $\psi$  is close

to  $\pi$  is considered similarly). Then we have the following estimate of the integral in (24)

$$\begin{aligned}
&\leq C \int_{-\varepsilon_1|\rho-1|^{-1}}^{\varepsilon_2|\rho-1|^{-1}} \frac{d\tau}{(|\tau|+1)\sqrt{||\rho-1|\tau+\cos\psi-1|}} = \\
&= \frac{C}{\sqrt{|\rho-1|}} \int_{-\varepsilon_1|\rho-1|^{-1}}^{\varepsilon_2|\rho-1|^{-1}} \frac{d\tau}{(|\tau|+1)\sqrt{\left|\tau + \frac{\cos\psi-1}{|\rho-1|}\right|}} \leq \frac{C}{\sqrt{|\rho-1|}}. \quad (25)
\end{aligned}$$

Thus, we obtain the estimate

$$\int_{\mathbb{S}^1} |K(\rho, \omega, \psi)| d\omega = O(|\rho-1|^{-1/2}) \text{ as } \rho \rightarrow 1.$$

Similarly, we obtain

$$\int_{\mathbb{S}^1} |K(\rho, \omega, \psi)| d\psi = O(|\rho-1|^{-1/2}).$$

These estimates and the Schur test [9] give the desired norm estimate of the integral operator

$$\|K(\rho)\| = O(|\rho-1|^{-1/2}).$$

The proof of the lemma is now complete. □

**Lemma 4.2.** *The operator-function  $\widehat{K}(p)$  (see (22)) enjoys the following properties*

1) *it is holomorphic for all  $p$  in the vertical strip*

$$\{-2 < \operatorname{Re} p < 1\} \subset \mathbb{C};$$

2) *it ranges in integral operators on  $\mathbb{S}^1$  with smooth kernel;*

3) *as  $\operatorname{Im} p \rightarrow \infty$  in the above described vertical strip, we have  $\|\widehat{K}(p)\| \rightarrow 0$ .*

*Proof.* Indeed, for all  $p$  in the vertical strip the integral (21) converges, since the integral of norms

$$\int_0^\infty |\rho^{p-1}| \cdot \|K(\rho)\| d\rho$$

is absolutely convergent by the estimates (22). The remaining statements of the lemma follow from well-known properties of the Mellin transform. □

We are now ready to describe the structure of the trace (13). Namely, we showed that this trace is localized at the fixed point and after applying Fourier and then Mellin transform in the radial variable in the dual space, the operator reduces to an operator of multiplication by the function  $\widehat{K}(p)$ . Gathering these transformations, we obtain a representation of the trace (13) in the form

$$\Delta_X^{-1/2} \chi \mathcal{F}^{-1} \chi' \mathcal{M}_{p \rightarrow \rho}^{-1} \widehat{K}(p) \mathcal{M}_{\rho \rightarrow p} \chi' \mathcal{F} \chi : H^s(X) \longrightarrow H^{s+1}(X). \quad (26)$$

Here  $\mathcal{F}$  is the Fourier transform,  $\mathcal{M}_{r \rightarrow p}$  is the Mellin transform in the radial variable in the dual space, while the cut-off functions  $\chi, \chi'$  are written to obtain a bounded operator in the corresponding function spaces. In more detail,  $\chi$  is a smooth function on  $X$  equal to zero outside a small neighborhood of zero and is identically equal to one in a small neighborhood of zero, while  $\chi'$  is a function in the dual space identically equal to one at infinity and zero in a neighborhood of zero. The trace (13) and the operator (26) are equal up to compact summands by the locality principle.

**Example 3** On the product  $\mathbb{R}_{x,z}^2 \times \mathbb{S}_y^1$ , we consider the action

$$g_\varphi(x, y, z) = (x \cos \varphi + z \sin \varphi, y + \varphi, -x \sin \varphi + z \cos \varphi), \quad \varphi \in \mathbb{S}^1$$

of the group  $\mathbb{S}_\varphi^1$  (this action consists of screw motions: shifts along  $y$  by  $\varphi$  and rotations in the  $XOZ$ -plane by angle  $\varphi$ ).

Let us study the trace of the  $G$ -operator

$$D = \Delta^{-1} \int_{\mathbb{S}^1} T_\varphi d\varphi, \quad \text{where } T_\varphi u(x, y, z) = u(g_\varphi^{-1}(x, y, z)),$$

on the submanifold equal to the horizontal coordinate plane:  $X = \{z = 0\}$ .

By the localization theorem the trace

$$i^!(D) = i^* \left( \Delta^{-1} \int_{\mathbb{S}^1} T_\varphi d\varphi \right) i_* : H^s(X) \rightarrow H^{s+1}(X), \quad s \in (-1, 0), \quad (27)$$

is localized at the submanifold  $X_{\mathbb{S}^1} = \{x = z = 0\} \subset X$  equal to the the  $OY$ -axis, about which we make rotations.

Then we represent  $H^s(X)$  as the space of sections of a Hilbert bundle over  $X_{\mathbb{S}^1}$  with fiber  $H^s(\mathbb{R})$  and we denote this bundle by  $\mathcal{H}^s(X_{\mathbb{S}^1})$ .

We represent the shift operator as the composition

$$T_\varphi = T'_\varphi T''_\varphi$$

of the shift  $T'_\varphi$  along  $X_{\mathbb{S}^1}$  and a rotation  $T''_\varphi$  in the  $XOZ$ -plane.

The structure of the trace (27) is described in the following proposition.

**Proposition 4.3.** *The trace (27) is a  $G$ -operator with operator-valued symbol on  $X_{\mathbb{S}^1}$  modulo compact summands. More precisely, the trace can be written as*

$$i^!(D) = \int_{\mathbb{S}^1} (i^* \Delta^{-1} T''_\varphi i_*) T'_\varphi d\varphi : \mathcal{H}^s(X_{\mathbb{S}^1}) \rightarrow \mathcal{H}^{s+1}(X_{\mathbb{S}^1}), \quad (28)$$

where the operator in brackets is a family of pseudodifferential operators on  $X_{\mathbb{S}^1}$  with operator-valued symbols. Moreover, the operators in the family continuously depend on  $\varphi$  in operator norm for all  $\varphi \neq 0, \pi$  and the norms of the operators and their symbols are uniformly bounded for all  $\varphi$ .

**Remark 4.1.** Note that pseudodifferential operators with operator-valued symbols of nonzero order were introduced in [10].

**Remark 4.2.** The operator family  $i^* \Delta^{-1} T''_{\varphi} i_*$  is not norm continuous at  $\varphi = 0$  and  $\pi$ . This is easy to see, since for small  $\varphi \neq 0$  the corresponding operator is localized at  $X_{\mathbb{S}^1} \subset X$ , while for  $\varphi = 0$  it is a pseudodifferential operator and, hence, is localized on the entire submanifold  $X$ . Similarly, one shows that there is no limit in norm as  $\varphi \rightarrow \pi$ .

*Proof.* A direct computation shows that (28) holds. Let us show that the operator in round brackets in (28) is a pseudodifferential operator with operator-valued symbol on  $X_{\mathbb{S}^1}$ . Indeed, writing this operator as

$$T''_{\varphi} i_{\varphi}^* \Delta^{-1} i_*,$$

where  $i_{\varphi} : g_{\varphi} X \rightarrow \mathbb{R}^2 \times \mathbb{S}^1$  stands for the shift of the initial submanifold  $X$  by an element  $g_{\varphi}$ , one can show that this operator is a  $\psi$ DO, since the factor  $T''_{\varphi}$  acts as identity along the base  $X_{\mathbb{S}^1}$ , while  $i_{\varphi}^* \Delta^{-1} i_*$  is a translator and, as shown in [11], is a  $\psi$ DO.

The boundedness of norms of these operators follows from their definition. To prove the norm boundedness of the operator family  $i^* \Delta^{-1} T''_{\varphi} i_*$ , and also obtain uniform boundedness of the symbols, we calculate the symbol (as an operator in the space dual with respect to the Fourier transform of functions depending on  $\xi, \eta$ ). This operator is equal to

$$\begin{aligned} u(\xi, \eta) \mapsto \int_{\mathbb{R}} \frac{u(\xi \cos \varphi - \zeta \sin \varphi, \eta) d\zeta}{\xi^2 + \eta^2 + \zeta^2} &= \int_{\mathbb{R}} \frac{u(z, \eta) dz}{|\sin \varphi| \left( \xi^2 + \eta^2 + \left( \frac{\xi \cos \varphi - z}{\sin \varphi} \right)^2 \right)} = \\ &= |\sin \varphi| \int_{\mathbb{R}} \frac{u(z, \eta) dz}{\xi^2 - 2\xi z \cos \varphi + z^2 + \eta^2 \sin^2 \varphi}. \end{aligned} \quad (29)$$

Here we made the change of variable  $z = \xi \cos \varphi - \zeta \sin \varphi$  in the integral. We denote the latter symbol by  $A_{\varphi}(\eta)$ . It smoothly depends on  $\eta$  and is *twisted homogeneous* (e.g., see [10]) with respect to this variable:

$$A_{\varphi}(\lambda \eta) = \lambda^{-1} \varkappa_{\lambda}^{-1} A_{\varphi}(\eta) \varkappa_{\lambda}, \quad \text{for all } \lambda > 0,$$

where  $\varkappa_{\lambda} f(z) = f(\lambda z)$  denotes the action of the group  $\mathbb{R}_+$  of dilations.

It remains to show that the symbol remains bounded as  $\varphi \rightarrow 0$ . By twisted homogeneity and unitarity of the group  $\varkappa_{\lambda}$ , we assume that  $\eta = 1$  and obtain

$$A_{\varphi}(1)u = |\sin \varphi| \int_{\mathbb{R}} \frac{u(z) dz}{(\xi - z \cos \varphi)^2 + \sin^2 \varphi \langle z \rangle^2}.$$

Here we use the notation  $\langle x \rangle = (1 + x^2)^{1/2}$ .

The commutative diagram

$$\begin{array}{ccc}
L^2(\mathbb{R}_z, \langle z \rangle^{2s}) & \xrightarrow{A_\varphi(1)} & L^2(\mathbb{R}_z, \langle z \rangle^{2(s+1)}) \\
\langle z \rangle^s \downarrow & & \langle z \rangle^{s+1} \downarrow \\
L^2(\mathbb{R}_z) & \longrightarrow & L^2(\mathbb{R}_z)
\end{array} \tag{30}$$

implies that the norm of  $A_\varphi(1)$  is equal to the  $L^2$ -norm of the operator  $\langle \xi \rangle^s A_\varphi \langle z \rangle^{-s}$ . So, it remains to estimate the norm of the integral operator in  $L^2(\mathbb{R})$ :

$$u(z) \mapsto \int_{\mathbb{R}} \frac{\langle \xi \rangle^{s+1} |\sin \varphi| \langle z \rangle^{-s} u(z) dz}{(\xi - z \cos \varphi)^2 + \sin^2 \varphi \langle z \rangle^2}.$$

The kernel of this integral operator is denoted by  $K(\xi, z)$ . To estimate the norm of this integral operator, we use the Schur test. To this end, let us obtain the uniform boundedness of the integrals

$$\int_{\mathbb{R}} |K(\xi, z)| dz, \quad \int_{\mathbb{R}} |K(\xi, z)| d\xi. \tag{31}$$

Let us estimate the first integral (the second is estimated similarly).

We have

$$\begin{aligned}
\int_{\mathbb{R}} |K(\xi, z)| dz &= \int_{\mathbb{R}} \frac{\langle \xi \rangle^{s+1} |\sin \varphi| \langle z \rangle^{-s} dz}{(z - \xi \cos \varphi)^2 + \sin^2 \varphi \langle \xi \rangle^2} = \int_{\mathbb{R}} \frac{dt}{t^2 + 1} \langle \xi \rangle^s \left( \langle \xi \cos \varphi + |\sin \varphi| \langle \xi \rangle t \rangle \right)^{-s} = \\
&= \int_{\mathbb{R}} \frac{dt}{t^2 + 1} \left( \frac{1}{\langle \xi \rangle} + \left( \frac{\xi}{\langle \xi \rangle} \cos \varphi + |\sin \varphi| t \right)^2 \right)^{-s/2} \leq \\
&\leq \int_{\mathbb{R}} \frac{dt}{t^2 + 1} (1 + (1 + |t|)^2)^{-s/2} < \infty. \tag{32}
\end{aligned}$$

Here in the second equality we made the change of variable  $z = \xi \cos \varphi + |\sin \varphi| \langle \xi \rangle t$  in the integral; while the first inequality follows from the fact that  $x^{-s}$  is increasing for  $s < 0$ ; the latter integral converges, since  $2 + s < 1$ .

Estimates of the norm of the second integral in (31) are obtained along the same lines.

So, by the Schur test the norm of the symbol is uniformly bounded. The continuity with respect to the operator norm follows from the fact that, as is easy to see, the symbol is differentiable with respect to the parameter  $\varphi$  if  $\sin \varphi \neq 0$ . Hence, the operator with this symbol continuously depends on this parameter in the operator norm.  $\square$



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